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Strategies in the secretary problem

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Dedicated to the memory of Ellis Kolchin
dear friend and greatly admired colleague.

1. **An anecdotal, early history.** A problem that has come to be known as "The Secretary Problem" is the focus of this note. My own acquaintance with the problem goes back to 1953 and a faculty club lunch table at Columbia University. At the lunch in question (Ellis Kolchin was present), one of the older physicists, having just returned from a consulting trip to Washington, remarked, "The boys in Washington are wondering whether there is a strategy for deciding which one of numbered balls in a box has the largest number as you draw them out." Of course, a quick comment such as that is sufficiently ambiguous to require a few minutes of clarification. What emerged was the following:

Problem. *A box contains a specified number n of balls each labeled with an integer, positive or negative, and no two with the same integer. The integers have their standard ordering. A player can cause a single ball to be ejected from the box by pressing a button, but has no information about the integer labeling of the balls until, of course, a given ball is ejected and available for inspection. The player does know n. As each ball is ejected, the player must decide if it is labeled with the largest of all the labeling integers. A win occurs just in the case where the player identifies the largest labeling as the corresponding ball is ejected.*

Now the very first thought most people have on hearing this problem, and that included all of us at the lunch table, is that you might as well eject one ball and name its labeling the largest; you have nothing else to go on. And so declared one of the young physicists at the table. The probability of winning was no better than $\frac{1}{n}$ for a purported "strategy." Prodded by the caution mathematicians learn so painfully, I demurred, "Perhaps a little more thought might show us a better strategy." He challenged, "What possible strategy is there?" At this point, I reached out quickly, somewhat wildly, and as it turns out, very luckily. "We might reject the first ball, out of

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hand," I said, "and then choose the next larger one ejected." John Tate, one of the mathematicians at that table, reached for a nearby paper napkin and began writing on it. "Let's try that when n is 3," he said. Of course, we can assume that the balls are in a single file tube and that we roll them out the open end one after the other according to their order in the tube. So each arrangement produces an ordering on which the strategy can be tested to see whether or not it produces a win. For this purpose, we may also assume that the balls are numbered 1, 2, and 3. John Tate's quick check showed that the strategy produces a win in three of the six cases; much better than $\frac{1}{3}$. The young physicist remarked that this result was surely an anomaly of the small value of n (as, in part, it is) and that, asymptotically (with n), no strategy could produce better than a $\frac{1}{n}$ chance of winning. I must confess that that made appealing "physical sense" to me.

The next day at lunch, both John Tate and I reported that if you reject $1/e$ of the n balls and choose the next higher labeled ball to appear, you win $1/e$ of the time — all this for large n (and, of course, to the nearest integer). Apparently, he, as I, at home that evening, had made the obvious generalization of the originally suggested strategy and checked for the maximum.

My argument went as follows. If we make no use of the specific labeling information (honor system!), we may assume that the balls are labeled 1 through n and refer to them by this label. We may also assume that a game is specified by a permutation of the n balls and that they are ejected from the box in order of their position (but not their labeling!). If, in a particular game, (the ball labeled) n occupies the j th position, the game ends at the j th draw. What the player does, depends on the labeled balls and their ordering in the "initial segment" of length $j-1$. With n in position j, there are $\binom{n-1}{j-1}$ sets of $j-1$ balls that can form this $j-1$ initial segment. If we have decided, as our strategy, to reject k balls and then to choose the next larger labeling to appear, with n in the j th position, there is a loss if $j \leq k$. If $k < j$, then there is a win with this strategy precisely when the largest labeled ball of the set of $j-1$ balls in the initial $j-1$ segment is among the k balls rejected, the first k balls. (In that position, it forces us to wait until we reach the ball labeled n in position j and to declare that ball the largest. Out of that position, and between the k th and j th positions, if it is reached, it is sure to be declared the largest, in the present strategy, causing a loss.) With the largest of the given set of $j-1$ balls in a definite one of the first k positions, there are $(j-2)!$ arrangements of the remaining $j-2$ balls of the initial $j-1$ segment, and, for each of these, $(n-j)!$ arrangements of the $n-j$ balls in the final $n-j$ segment, each of which produces a win. Thus there are $k(j-2)!(n-j)!$ wins with the given set of $j-1$ balls as the initial $j-1$ segment and n in position j. There are $k\binom{n-1}{j-1}(j-2)!(n-j)!$ wins with n in the j th position and k less than j. The total number of wins W_k with this strategy satisifies $W_k = k\sum_{j=k+1}^{n}\binom{n-1}{j-1}(j-2)!(n-j)!$. Thus

$$W_k = k\sum_{j=k+1}^{n}\frac{(n-1)!(j-2)!}{(j-1)!(n-j)!}(n-j)! = k\sum_{j=k+1}^{n}\frac{(n-1)!}{j-1}.$$

Of the $n!$ possible games, the fraction that are wins is $\frac{k}{n}\sum_{j=k+1}^{n}\frac{1}{j-1}$. Now,

$$\sum_{j=k+1}^{n}\tfrac{1}{j-1} = \tfrac{n}{k}\cdot\tfrac{1}{n} + \tfrac{n}{k+1}\cdot\tfrac{1}{n} + \cdots + \tfrac{n}{n-1}\cdot\tfrac{1}{n},$$

which is the Riemann approximating sum to $\int_{\frac{k}{n}}^{1}\frac{dx}{x}$ $(= -\log\frac{k}{n})$ for the interval $[0,1]$ partitioned into n equal parts and evaluating $\frac{1}{x}$ at the partition points $\frac{k}{n}, \frac{k+1}{n}, \ldots \frac{n-1}{n}$. If we let x be $\frac{k}{n}$, the fraction of the balls rejected, the fraction of the total number of games that are wins is (asymptotically) $-x\log x$. The maximum occurs when x is $1/e$ and equals $1/e$. A more detailed account of this maximization, especially as it relates the discrete to the continuous, appears in Section 4 in connection with the identification of the best "deterministic" strategy.

The strategies, reject k balls and choose the next highest, were the first ones that had come to mind; there were certainly other strategies that made sense. I said so. The young physicist commented that the $1/e$ strategy was probably the best. John Tate said much more definitively that it was surely the best. Starting from our initial assessment, winning $1/e$ of the time certainly did seem splendid. "What better strategies could there be," was asked. I ventured, somewhat quickly and wildly again, that we could reject k draws and choose the second highest draw to appear. John Tate reaffirmed his feeling that there could not be anything better than the $1/e$ strategy.

That weekend, I pondered what might be meant by a "strategy" and proved that Tate was right. The substance of that argument (in more polished form) is given in Section 4. It and the discussion of strategies is the main point to this article.

Roughly a year after the events just noted, the following short argument occurred to me for seeing that the strategy of rejecting the first fraction of the draws and accepting the following high draw has serious potential, independently of n. We partition an arrangement of $\{1,\ldots,n\}$ (a "game") into four (approximately) equal segments, the first through fourth "quarters" of the game, and track the location of the numbers $n-1$ and n. There are sixteen (roughly) equally probable locations of these numbers in the four quarters. In three of these, when $n-1$ is in the first quarter and n is not, a win is guaranteed with the strategy that rejects the first quarter and chooses the next highest. Of course, there are more wins with this strategy, but we are assured wins in 3/16 of the games with no further calculation and no dependence on the number n.

Bernard Koopman (our Department Chair at that time) had invited me to prepare and publish these considerations in the fledgling operations research journal. I declined, with the press of other mathematical work as my reason. My thanks are due to Carsten Thomassen for convincing me that I should not let "another forty years" pass without making this generally available.

2. **What is a strategy?** This is a delicate and complicated question. How do we rule out cups of coffee and horoscopes as components of our potential strategies? We must strive for some semblance of rationality in deciding what a strategy should be, but if we are too restrictive, we may eliminate some approach that is valuable, though not clearly so. This is a situation not unlike the one confronting the mathematicians of the Eighteenth Century — plagued by an undefined or ill-defined (and too restrictive) notion of 'function.'

At the very least, presented with a single game, that is, an arrangement of the labeled balls in the order in which they will be ejected, a strategy should specify when the game is ended — whatever goes into that specification, rejecting a given number of balls, horoscopes, *etc.* In the broadest sense, then, a strategy S is a mapping from a total ordering of the labeled balls to some initial segment of that total ordering. The terminal element of that initial segment is understood to be the one declared by the player to have the largest labeling (for better or worse). We are not insisting that a particular strategy is sensible or even practicable.

With this broad definition of strategy, of course there is a better strategy than the $1/e$ strategy: map each total ordering to the initial segment that ends with n. This might be called "the omniscience strategy." It produces a win every time and should definitely be preferred by total clairvoyants. Seeking strategies that are practicable and rational, we can probably agree that, when we declare the terminal element of the (ordered) set of ejected balls to be the largest in a given game (that is, total ordering of the n balls), then we should do so again when that initial segment appears as an initial segment in another game. Since the integers of the labeling are supposed to be totally unknown to the player, we must go even further in seeking rational strategies and require that, when the initial segment of the same length in the second game is order isomorphic to the initial segment that ends the first game, then it ends the second game. We shall call such strategies *deterministic* — the action at a particular draw, acceptance or rejection, is completely determined by the available (order-theoretic) information. In Section 4, the deterministic strategies are described more fully; it is proved (Theorem A) that the "$1/e$ strategy" is the best (produces the most wins) among them.

If we have reached the fifth draw in a particular game, given that the number drawn is larger than the four preceding, the question of whether or not to end the game on the fifth draw seems to have little to do with the first drawn number being larger than the second or the second being larger than the first. It would seem that, having ended the game on the fifth draw once (when the element drawn is larger than the four preceding), we should end the game on the fifth draw each time the element drawn is larger than the four preceding. These considerations lead to the definition of *position strategies*. They are described by listing those positions at which the game is allowed to end and by insisting that the game does end at the first of those positions at which the element drawn is larger than all the preceding. To be sure, these position strategies are deterministic. The position strategies are analyzed, in detail, in the next section.

The argument for considering only the position strategies among the deterministic strategies is a cogent one. On the other hand, the strategy that ends a game the second time we draw a number larger than all the preceding numbers drawn after rejecting, say, the first $n/4$ numbers drawn, certainly has some credibility as a sensible strategy. It is not a position strategy, though it is a deterministic strategy. If we are searching for the "best" strategy within the realm of "reasonable" strategies (accordingly, ruling out the "omniscience strategy"), we must take into account the full range of deterministic strategies.

3. **Position strategies.** A strategy S that ends a game only in one of a prescribed set of positions $j_1 < j_2 < \cdots < j_r$, and that *does* end the game if it reaches one of those positions and the number at that position is greater than all the numbers in positions preceding it, will be called a *position strategy*, the position strategy *based on positions* $j_1, \ldots, j_r$. Each j_h will be called a *position for the strategy* (or a *strategy position*). In this section, we shall analyze position strategies and show that the position strategy based on positions $j_1, \ldots, j_r$, in an n-element game, produces fewer winning games

than the position strategy based on positions $n-r+1,\ldots,n$. We shall prove this in two ways. For the first proof, we construct an injective mapping from the set of games won in one strategy into the set of games won in the other. For the second proof, we develop precise formulas for the number of games ending in each position for the strategy and for the number of games won in each such position.

Suppose S is the position strategy that ends games at positions $j_1 < j_2 < \cdots < j_s$ and suppose that $j_k < j'_k < j_{k+1}$. Let S' be the strategy that ends games at $j_1, j_2, \ldots, j_{k-1}, j'_k, j_{k+1}, j_{k+2}, \ldots, j_s$ (that is, j_k has been "shifted further to the right" in the strategy S to produce the strategy S'). We shall show that there are more wins in the strategy S' than in S. Note that if a game ends in one of the positions $j_1,\ldots,j_{k-1}$ in S, it ends at that position in S'; it results in a win in either strategy if and only if n occupies that position. Thus, denoting by '$w_S(j)$' the number of wins in position j in the strategy S,

$$w_S(j_1) = w_{S'}(j_1), \ldots, w_S(j_{k-1}) = w_{S'}(j_{k-1}).$$

At the same time, a win occurs in position j_k in strategy S if and only if the game ends at j_k and n is in position j_k. The game that ends at j_k with n in position j_k in S will end at j'_k in S' and result in a win in that same arrangement, with the elements at positions j_k and j'_k interchanged, for we have not ended the game in positions $j_1,\ldots,j_{k-1}$ in this transposed game (or it would have ended there before the transposition) and it cannot end before j'_k in S' having passed j_{k-1}. Of course, with n in position j'_k, it does end there and results in a win. Thus the transposition of the elements in positions j_k and j'_k is a one-to-one mapping of the games onto themselves that carries the winning games in position j_k in S onto the winning games in position j'_k in S'. Hence $w_S(j_k) = w_{S'}(j'_k)$.

If n occupies position j_m, the game is won there if it does not end before. In strategy S, if the game has not ended in position j_k, the element in that position is not greater than all the preceding elements. If the element in position j_k is interchanged with the element in position j'_k, then the game does not end at j'_k, and cannot end before, in strategy S'. Since the same elements precede those at positions $j_{k+1},\ldots,j_{m-1}$ in the transposed game as in the original, the transposed game does not end at these positions in S'. Thus the transposition converts a winning game in position j_m in strategy S to a winning game in position j_m in strategy S'. The reverse is not true for positions $j_{k+1},\ldots,j_s$; a winning game in strategy S' need not be a winning game after the transposition, in strategy S. The element at position j'_k in the game won at position j_m ($m > k$) may not end the game at position j'_k (and, so, not block the win at position j_m) because some element in a position strictly between j_k and j'_k is larger than it. But when this same element has been moved down to ("interchanged with" the element at) position j_k it no longer competes with that larger element (in a position between j_k and j'_k). It may very well be larger than the elements in positions $1,\ldots,j_k - 1$ and now end the transposed game at position j_k preventing the win in position j_m. Thus transposing the elements in positions j_k and j'_k is a one-to-one mapping of the set of games onto itself that maps the set of games won in strategy S (strictly) into the set of games won in strategy S'. There are more games won in strategy S' than in strategy S.

The lesson to be drawn from the preceding discussion is that a better strategy is obtained by moving an ending position for the given strategy S across a gap, to the right, to the position just preceding the following ending position for S. Pursued, relentlessly, this "improvement" process results in a strategy whose ending positions form an interval with terminal position and total number of positions the same as for S. If the strategy S does not include the n th position as an ending position, then adding it as an ending position certainly improves the strategy: if we reach the last position with strategy S and encounter an element larger than all the preceding (hence, the largest), then of course we must accept it and win. Doing this does not, in the least, diminish our chance of winning in an earlier position under strategy S. Combined with what we have learned, this tells us that a best strategy must be a "segment strategy," reject the first given number of elements drawn and then choose the first element larger than all the preceding. The discussion of Section 1 indicates, now, that the segment strategy in which the first $1/e$ draws are rejected (that is, exclude the first $1/e$ positions from the set of strategy positions) is the best strategy.

Given an ordered r-tuple $\langle j_1, j_2, \ldots, j_r\rangle$ of numbers in $\{1,\ldots,n\}$, where $j_1 < j_2 < \cdots < j_r$, we develop a formula for the number of permutations of $\{1,\ldots,n\}$ in which the number occupying position j_m is larger than all the numbers preceding it but this is not the case for the numbers occupying positions j_1, j_2, $\ldots$, j_{m-1}. There are $\binom{n}{j_1}$ sets of numbers from among $\{1,\ldots,n\}$ that can occupy the first j_1 positions. for each such set there are $(n - j_1)!$ arrangements of the remaining $n - j_1$ elements in the last $n - j_1$ positions. For the given set of j_1 elements, with the largest of this set in the j_1 position there are $(j_1 - 1)!$ arrangements of the remaining elements. For this given set of j_1 elements there are $(j_1 - 1)!(n - j_1)!$ arrangements with the element in position j_1 larger than the preceding elements. Thus there are

$\binom{n}{j_1}(j_1-1)!(n-j_1)!$ arrangements of $\{1,\dots,n\}$ with the number in position j_1 greater than all the preceding numbers, that is, $n!/j_1$ such arrangements.

Similarly, there are $n!/j_2$ such arrangements for position j_2; a certain number of these have the number in position j_1 larger than all the numbers preceding it, and we want to exclude these. For each set of j_2 numbers from $\{1,\dots,n\}$, with the largest of the set in position j_2, the remaining j_2-1 elements of this set can be arranged so that the number in position j_1 is larger than all the preceding numbers in $(j_2-1)!/j_1$ ways and for each of these ways there are $(n-j_2)!$ arrangements of the numbers not in the given set of j_2 numbers. Thus there are

$$\tfrac{n!}{j_2}-\tbinom{n}{j_2}\tfrac{(j_2-1)!}{j_1}(n-j_2)!=\tfrac{n!}{j_2}(1-\tfrac{1}{j_1})$$

arrangements of $\{1,\dots,n\}$ such that the number in position j_2 is larger than all the preceding numbers and the number in position j_1 is not larger than all the numbers preceding it.

Again, the number of arrangements of $\{1,\dots,n\}$ that have the number in position j_3 larger than all the preceding numbers is $\frac{n!}{j_3}$. When we have selected a set of j_3 numbers from $\{1,\dots,n\}$, there are $\frac{(j_3-1)!}{j_2}(1-\frac{1}{j_1})$ arrangements of them in which the largest number is in position j_3, the number in position j_2 is larger than all the preceding numbers and this is not the case for the number in position j_1, from the argument of the preceding paragraph. There are $\frac{(j_3-1)!}{j_1}$ arrangements of the j_3 numbers in the given set with the largest number in position j_3 and the number in position j_1 larger than the numbers in all the positions preceding it. Thus there are

$$(j_3-1)!-\tfrac{(j_3-1)!}{j_2}(1-\tfrac{1}{j_1})-\tfrac{(j_3-1)!}{j_1}=(j_3-1)!(1-\tfrac{1}{j_2}+\tfrac{1}{j_1j_2}-\tfrac{1}{j_1})$$

arrangements of a given set of j_3 numbers chosen from $\{1,\dots,n\}$ in which the number in position j_3 is larger than the number in all the preceding positions, but this is not the case for the numbers in positions j_1 or j_2. There are

$$\tbinom{n}{j_3}(n-j_3)!(j_3-1)!(1-(\tfrac{1}{j_1}+\tfrac{1}{j_2})+\tfrac{1}{j_1j_2})=\tfrac{n!}{j_3}(1-(\tfrac{1}{j_1}+\tfrac{1}{j_2})+\tfrac{1}{j_1j_2})$$

arrangements of $\{1,\dots,n\}$ in which the number in position j_3 is larger than all the preceding numbers and this is not the case for the numbers in positions j_1 or j_2.

The general formula begins to emerge. Define p_h^k to be the sum of all the reciprocals of products of $h-1$ distinct numbers from $\{j_1,j_2,\dots,j_{k-1}\}$, when

$h \leq k$, and let p_1^k be 1 for all k. We assert that the numbers of arrangements of $\{1,\ldots,n\}$ in which the number in position j_k is larger than the number in all the preceding positions and this is not the case for the number in any of the positions $j_1, \ldots, j_{k-1}$ is

$$(1) \qquad \tfrac{n!}{j_k}(p_1^k - p_2^k + p_3^k - \cdots + (-1)^{k+1}p_k^k).$$

We prove this by induction on m. So we assume that (1) determines the number of arrangements of $\{1,\ldots,n\}$ in question for all n when there are fewer than m positions $j_1, j_2, \ldots, j_m$ under consideration (that is when $k = 1,\ldots,m-1$).

For each set of j_m numbers from $\{1,\ldots,n\}$, there are $(j_m-1)!$ arrangements in which the largest of these numbers occupies the position j_m. We must exclude from these arrangements those in which the number occupying any one of positions $j_1, \ldots, j_{m-1}$ is laıger than all the numbers preceding it. First, we exclude $\frac{(j_m-1)!}{j_1}$ arrangements in which the number in position j_1 is larger than all the preceding, then the $\frac{(j_m-1)!}{j_2}(1-\frac{1}{j_1})$ arrangements in which the number in position j_2 is larger than all the preceding but the number in position j_1 is not larger that those preceding (for any such arrangement has already been excluded in the $\frac{(j_m-1)!}{j_1}$ arrangements described). Continuing in this way, and using the inductive hypothesis, we exclude the

$$\tfrac{(j_m-1)!}{j_k}(p_1^k - p_2^k + \cdots + (-1)^{k+1}p_k^k)$$

arrangements in which the number in position j_k is larger than all the numbers preceding it and that is not the case for any of the numbers in positions $j_1, \ldots, j_{k-1}$, successively as k assumes the values $1, \ldots, j_{m-1}$. In total, then, we exclude

$$(2) \qquad (j_m-1)! \sum_{k=1}^{m-1} \tfrac{1}{j_k}(p_1^k - p_2^k + \cdots + (-1)^{k+1}p_k^k)$$

arrangements of the given set of j_m elements chosen from $\{1,\ldots,n\}$ from the $(j_m-1)!$ arrangements in which the largest number of the set is in the j_m th position.

We note that $\sum_{k=h}^{m-1} \frac{1}{j_k}p_h^k$ is p_{h+1}^m. With $h \leq k$, if we sum the reciprocals of the products of $h-1$ distinct numbers chosen from among $\{j_1,\ldots,j_{k-1}\}$ (that is, form p_h^k) and multiply this sum by $\frac{1}{j_k}$, we arrive at the sums of the reciprocals of products of h distinct numbers from among $\{j_1,\ldots,j_k\}$, where one of the numbers is j_k. If we, now, sum these sums as k assumes the values

h, $h+1$, …, $m-1$, the result is the sum of the reciprocals of all products of h distinct numbers chosen from $\{j_1, \ldots, j_{m-1}\}$, that is, the result is p^m_{h+1}. It follows that (2) is equal to

$$(j_m - 1)!(p^m_2 - p^m_3 + \cdots + (-1)^m p^m_m) \tag{3}$$

and that the number of arrangements of the given set of j_m elements with the largest number in position j_m, and the number in each of the positions j_1, …, j_{m-1} is not larger than all the numbers preceding it, is

$$(j_m - 1)! - (j_m - 1)!(p^m_2 - p^m_3 + \cdots + (-1)^m p^m_m) = (j_m - 1)! \sum_{h=1}^{m} (-1)^{h+1} p^m_h.$$

Each of these arrangements carries with it $(n - j_m)!$ arrangements of $n - j_m$ numbers not in the set of j_m numbers chosen from $\{1, \ldots, n\}$. Finally, there are $\binom{n}{j_m}$ choices of the set of j_m numbers. Thus there are

$$\begin{aligned} &\tbinom{n}{j_m}(n - j_m)!(j_m - 1)! \sum_{h=1}^{m} (-1)^{h+1} p^m_h \\ &\qquad = \tfrac{n!}{j_m}(p^m_1 - p^m_2 + \cdots + (-1)^{m+1} p^m_m) \end{aligned} \tag{4}$$

arrangements of $\{1, \ldots, n\}$ with the number in position j_m larger than all the numbers preceding it and this not being the case for the numbers in any of the positions j_1, j_2, …, j_{m-1}. The inductive step has been established, and the validity of the formula (1) follows.

Using (1), we shall, now, derive a formula for the number of arrangements of $\{1, \ldots, n\}$ in which n occupies position j_m and none of the numbers in positions j_1, …, j_{m-1} is larger than all the numbers preceding it. Note, for this, that in the inductive step of the preceding argument, if we insist that n be in the set of j_m numbers chosen from $\{1, \ldots, n\}$, then there are $\binom{n-1}{j_m-1}$ instead of $\binom{n}{j_m}$ choices, n will, of necessity, be the largest number in this j_m-element set, and the remainder of the discussion is unaltered. Thus the formula we are deriving becomes

$$\begin{aligned} &\tbinom{n-1}{j_m-1}(n - j_m)!(j_m - 1)! \sum_{h=1}^{m} (-1)^{h+1} p^m_h \\ &\qquad = (n-1)!(p^m_1 - p^m_2 + \cdots + (-1)^{m+1} p^m_m). \end{aligned} \tag{5}$$

It is now time to recognize the expression appearing in (4) and (5) for what it is:

$$(6)\quad p_1^m - p_2^m + \cdots + (-1)^{m+1} p_m^m = (1-\tfrac{1}{j_1})(1-\tfrac{1}{j_2})\cdots(1-\tfrac{1}{j_{m-1}}) \quad (m \geq 2).$$

A few moments of thought on how the terms of the expansion of the right-hand side of (6) as a sum are formed and the definition of p_h^m is sufficient to establish this identity. We add a few lines of formal argument. Define $p_h(k_1, \ldots, k_r)$ to be the sum of the reciprocals of all products of $h-1$ distinct terms from among $\{k_1, \ldots, k_r\}$ and $p_1(k_1, \ldots, k_r)$ to be 1. Thus $p_h^m = p_h(j_1, \ldots, j_{m-1})$. Define $c_m(k_1, \ldots, k_r)$, with $m \geq 2$, to be

$$p_1(k_1, \ldots, k_{m-1}) - p_2(k_1, \ldots, k_{m-1}) + \cdots + (-1)^{m+1} p_m(k_1, \ldots, k_{m-1}).$$

Note that, when $2 \leq h \leq m-1$,

$$p_h(j_1, \ldots, j_{m-1}) = \tfrac{1}{j_1} p_{h-1}(j_2, \ldots, j_{m-1}) + p_h(j_2, \ldots, j_{m-1})$$

and that

$$p_m(j_1, \ldots, j_{m-1}) = \tfrac{1}{j_1} p_{m-1}(j_2, \ldots, j_{m-1}).$$

Thus

$$\begin{aligned}
&c_m(j_1, \ldots, j_r) \\
&= \sum_{h=1}^{m} (-1)^{h+1} p_h(j_1, \ldots, j_{m-1}) \\
&= \sum_{h=1}^{m-1} (-1)^{h+1} p_h(j_2, \ldots, j_{m-1}) - \tfrac{1}{j_1} \sum_{h=1}^{m-1} (-1)^{h+1} p_h(j_2, \ldots, j_{m-1}) \\
&= (1-\tfrac{1}{j_1}) c_{m-1}(j_2, \ldots, j_r) \\
&\ \vdots \\
&= (1-\tfrac{1}{j_1})(1-\tfrac{1}{j_2})\cdots(1-\tfrac{1}{j_{m-2}}) c_2(j_{m-1}, \ldots, j_r) \\
&= (1-\tfrac{1}{j_1})(1-\tfrac{1}{j_2})\cdots(1-\tfrac{1}{j_{m-2}})(1-\tfrac{1}{j_{m-1}}),
\end{aligned}$$

which establishes (6).

It is clear from (6) that $p_1^m - p_2^m + \cdots + (-1)^{m+1} p_m^m$ is positive when each $j_h > 1$ — though, this is clear, as well, from the interpretation of (4) and (5) as certain numbers of arrangements. Less clear is the fact that (5) increases if some j_k is replaced by a larger j'_k, where $1 \le k \le m-1$, though this is apparent from the product formula given by (6).

The meaning of the information about arrangements of $\{1, \dots, n\}$ given by (4) and (5) in terms of our position strategies is immediate. If S is the position strategy corresponding to positions $j_1, \dots, j_r$, then (4) is the formula for the number of games ending in position j_m in this strategy and (5) is the formula for the number of games won at position j_m in strategy S. As just noted, the shift from j_k to a larger j'_k increases the number of wins at positions $j_{k+1}, \dots, j_r$ (and leaves the number of wins in positions $j_1, \dots, j_{k-1}, j'_k$ unchanged). Thus this shift produces a strategy that leads to a greater number of wins than the original strategy. Again, this time from the formulas (4), (5), and (6), we can conclude that the position strategy corresponding to positions $n-r+1, \dots, n$ is better than (or at least as good as) S.

4. **Deterministic strategies.** In this section, we study the deterministic strategies, establishing various formulas and showing (Theorem A) that there is a unique deterministic strategy yielding the maximum number of wins (among the deterministic strategies) and that it is the position strategy: Reject the first n/e draws and choose the next draw larger than these.

Since the deterministic strategies are completely described in terms of order, their description is best given in order-theoretic terms. Toward this end, we let $\mathcal{N}$ be the set of ordered k-tuples, $k = 1, \dots, n$ of numbers from $\{1, \dots, n\}$ with distinct terms ($\langle 1,5\rangle$, $\langle 8,6,3\rangle$, and $\langle 1, \dots, n\rangle$ are members of $\mathcal{N}$, but $\langle 2,6,2\rangle$ is not, where $n \ge 8$). We refer to k as the *length* of a k-tuple b in $\mathcal{N}$ (and we denote this length by '$|b|$'), so that $\mathcal{N}$ is equipped with a "grading." We define the relation '$\sim$' on $\mathcal{N}$ as "order-isomorphism" of elements a $(= \langle a_1, \dots, a_k\rangle)$ and b $(= \langle b_1, \dots, b_k\rangle)$. Thus $a \sim b$ precisely when $a_j \le a_h$ if and only if $b_j \le b_h$ for all j and h in $\{1, \dots, k\}$, where the integers have their usual ordering. Of course, $\sim$ is an equivalence relation on $\mathcal{N}$. Let $\tilde{\mathcal{N}}$ be the set of equivalence classes of elements of $\mathcal{N}$ under $\sim$.

With a in $\mathcal{N}$, denote by '$\tilde{a}$' the equivalence class of a. Since equivalent elements of $\mathcal{N}$ have the same length, $\tilde{\mathcal{N}}$ inherits the grading of $\mathcal{N}$; let '$|\tilde{a}|$' denote the length of each of the elements in $\tilde{a}$. We refer to $|\tilde{a}|$ as the *length* of $\tilde{a}$.

Define '$a \le b$', for elements a and b of $\mathcal{N}$, to mean that a is an initial segment of b (so, for example, $\langle 3,1,2\rangle \le \langle 3,1,2,5,7\rangle$). The relation $\le$ is a partial ordering on $\mathcal{N}$. We denote by '$\le$', again, the relation inherited on $\tilde{\mathcal{N}}$ from $\le$ on $\mathcal{N}$; that is $\tilde{a} \le \tilde{b}$ precisely when $a \le b$ for some a in $\tilde{a}$ and b

in $\tilde{b}$. Of course, $\tilde{a} \leq \tilde{a}$ for each $\tilde{a}$ in $\tilde{\mathcal{N}}$. If $\tilde{a} \leq \tilde{b}$, then $a \leq b$ for some a in $\tilde{a}$ and b in $\tilde{b}$, whence $\tilde{a}$ has length not exceeding that of $\tilde{b}$. If, at the same time, $\tilde{b} \leq \tilde{a}$, then $\tilde{a}$ and $\tilde{b}$ have the same length. Thus a and b have the same length. Since a is an initial segment of b, $a = b$, whence $\tilde{a} = \tilde{b}$.

Suppose $\tilde{a} \leq \tilde{b}$ and $\tilde{b} \leq \tilde{c}$. Then there are a, b, b', and c, in $\tilde{a}$, $\tilde{b}$, $\tilde{b}$, and $\tilde{c}$, respectively, such that $a \leq b$ and $b' \leq c$. Let a have length j and let a_0 be the initial segment of b' of length j. Since $b \sim b'$ and a is the initial segment of b of length j, we have that $a \sim a_0$ so that $\tilde{a} = \tilde{a}_0$. Now, $a_0 \leq b' \leq c$, whence $\tilde{a} \leq \tilde{c}$. It follows that $\leq$ is a partial ordering on $\tilde{\mathcal{N}}$.

The deterministic strategies may be described in these order-theoretic terms simply as a subset S of $\tilde{\mathcal{N}}$ with certain properties, it being understood that the elements of S are the order-types of the initial segments of games that cause those games to end when they occur at a given draw. The key condition for a strategy to be deterministic translates to a property of S. If $\tilde{a}$ and $\tilde{b}$ are distinct elements of S, then $\tilde{a}$ and $\tilde{b}$ are not comparable (that is, we have neither $\tilde{a} < \tilde{b}$ nor $\tilde{b} < \tilde{a}$). We call a subset in which distinct elements are not comparable *disordered*. Thus the deterministic strategies correspond to subsets S of $\tilde{\mathcal{N}}$ that are disordered.

Another factor that needs consideration in our discussion of "rational" strategies is the avoidance of "deliberate" losses. It does not seem to make sense to end an n-element game before the n th draw at an element smaller than some previously drawn element. That would guarantee a loss and the associated order type could never produce a win. We prove that a strategy corresponding to a disordered subset S of $\tilde{\mathcal{N}}$ that includes a "deliberate" loss (some $\tilde{a}$, where $a = \langle a_1, \ldots, a_k \rangle$, $k < n$, and $a_k < a_j$ for some j in $\{1, \ldots, k-1\}$) can be improved to a strategy corresponding to a disordered set without such losses. For this purpose, define a *top* element of $\mathcal{N}$ as a k-tuple $\langle b_1, \ldots, b_k \rangle$ $(= b)$ such that $b_j \leq b_k$ for each j in $\{1, \ldots, k\}$. Of course, each member of $\tilde{b}$ is a top element of $\mathcal{N}$ if one of its members is. In this case, we call $\tilde{b}$ a *top* element of $\tilde{\mathcal{N}}$.

If S is a disordered subset of $\tilde{\mathcal{N}}$ and $\tilde{a}$, in S, is not a top element, with $|\tilde{a}|$ $(= k)$ less than n, order the numbers $1, \ldots, k$ so that the resulting ordered k-tuple $\langle a_1, \ldots, a_k \rangle$ $(= a)$ lies in $\tilde{a}$. Now let S' be S with $\tilde{a}$ replaced by $\tilde{a'}$, where $a' = \langle a_1, \ldots, a_k, n \rangle$. Note that S' is disordered, for if $\tilde{b} < \tilde{a'}$, with $\tilde{b}$ in S', then $\tilde{b} < \tilde{a}$ (since $\tilde{a}$ has been removed from S, $\tilde{b} \neq \tilde{a}$). As $\tilde{b}$ and $\tilde{a}$ are in S, this relation would contradict the assumption that S is disordered. On the other hand, if $\tilde{a'} < \tilde{c}$ with $\tilde{c}$ in S' (hence, in S), then $\tilde{a} < \tilde{c}$, again, contradicting the fact that S is disordered. Of course, the new element $\tilde{a'}$ corresponds to wins for S' while $\tilde{a}$ produces no wins for S. Since S and S'

differ only at the elements $\tilde{a}$ and $\tilde{a'}$, all other wins are shared by each of them. It follows that S' has more wins than S (and one fewer "deliberate" loss). Continuing in this way, we arrive at a strategy corresponding to a disordered subset S'' with all members top elements of $\tilde{\mathcal{N}}$, having more wins than S.

Definition. *A deterministic strategy is a disordered subset of $\tilde{\mathcal{N}}$ consisting of top elements.*

We establish some formulas in the present framework. They will be useful in showing that the best deterministic strategy is a position strategy. With k and h positive integers such that $k < h$ and $\tilde{a}$ in $\tilde{\mathcal{N}}$ of length k, we determine the number of top elements $\tilde{b}$ there are in $\tilde{\mathcal{N}}$ of length h such that $\tilde{a} < \tilde{b}$. Each such $\tilde{b}$ is the equivalence class of some arrangement of the numbers $1, \ldots, h$. How many such arrangements have their initial segment of length k in $\tilde{a}$, reserving h for the last element since $\tilde{b}$ is a top element? Each choice of k numbers from $\{1, \ldots, h-1\}$ can be arranged in just one way so that the resulting ordered k-tuple lies in $\tilde{a}$. There are, then, $(h-k-1)!$ arrangements of the remaining $h-k-1$ elements (exclusive of h) to yield top elements $\tilde{b}$ of $\tilde{\mathcal{N}}$ such that $\tilde{a} < \tilde{b}$. For distinct choices of k elements from $\{1, \ldots, h-1\}$, the corresponding sets of $(h-k-1)!$ top elements of $\tilde{\mathcal{N}}$ are disjoint (the arrangements of $1, \ldots, h$ are different for the different choices). Thus there are

$$\binom{h-1}{k}(h-k-1)! = \frac{(h-1)!}{k!} \tag{7}$$

top elements in our set $\tilde{\mathcal{N}}$ of length h greater than a given element $\tilde{a}$ in $\tilde{\mathcal{N}}$ of length k.

With $\tilde{b}$ a top element of length h in $\tilde{\mathcal{N}}$, each choice of h numbers from $\{1, \ldots, n\}$ including n (thus, a choice of $h-1$ numbers from $\{1, \ldots, n-1\}$) can be arranged in just one way so that the corresponding ordered h-tuple lies in $\tilde{b}$. Corresponding to each such choice, there are $(n-h)!$ arrangements of the remaining $n-h$ numbers. It follows that when $\tilde{b}$ is in a deterministic strategy S, there are

$$\binom{n-1}{h-1}(n-h)! = \frac{(n-1)!}{(h-1)!} \tag{8}$$

wins associated with $\tilde{b}$. Since there are $\frac{(h-1)!}{k!}$ such elements $\tilde{b}$ greater than a given element $\tilde{a}$ of length k less than h (from (8)), there are

$$\frac{(h-1)!}{k!}\frac{(n-1)!}{(h-1)!} = \frac{(n-1)!}{k!} \tag{9}$$

wins in position h that are eliminated if $\tilde{a}$ is part of a strategy.

Theorem A. *There is a unique deterministic strategy that produces more wins than any other deterministic strategy. It is a position strategy.*

Proof. Let S be a deterministic strategy and let s_h be the number of elements in S of length h. From (8), there are

$$\sum_{h=1}^{n} s_h \frac{(n-1)!}{(h-1)!} \tag{10}$$

wins with this strategy. For each element in S of length k, there are $\frac{(h-1)!}{k!}$ top elements of $\tilde{\mathcal{N}}$ of length h greater than it. None of these top elements are in S since S is deterministic (disordered). The number of top elements in $\tilde{\mathcal{N}}$ of length h is $(h-1)!$. Thus

$$0 \le s_h \le (h-1)! - \sum_{k=1}^{h-1} s_k \frac{(h-1)!}{k!} = (h-1)![1 - \sum_{k=1}^{h-1} \frac{s_k}{k!}]. \tag{11}$$

Note that two distinct elements of S cannot have some one element of $\tilde{\mathcal{N}}$ greater than each of them, for then they are comparable, while S is assumed to be disordered. For the preceding inequality, then, we are not counting, more than once, the same element of $\tilde{\mathcal{N}}$ of length h prohibited membership in S by virtue of being larger than some element in S of length less than h. Note, too, that subject to the inequality (11) and to the condition that the numbers s_h are integers, there is a deterministic strategy S that corresponds to these values s_h. To see this, observe that having chosen subsets $S_1, \ldots, S_{h-1}$ of $\tilde{\mathcal{N}}$ such that S_j has s_j elements each of length j and $\cup_{j=1}^{h-1} S_j$ is disordered, the inequality (11) is precisely the condition that there is a set S_h of s_h elements in $\tilde{\mathcal{N}}$ each of length h none of which is greater than any of the elements of $\cup_{j=1}^{h-1} S_j$. Then $\cup_{j=1}^{h} S_j$ is disordered. In this way we build the desired strategy S $(= \cup_{j=1}^{n} S_j)$ with s_h elements of length h for each h in $\{1, \ldots, n\}$.

Letting t_k be $\frac{s_k}{k!}$, we may rewrite (11) as

$$0 \le t_h \le \tfrac{1}{h}[1 - \sum_{k=1}^{h-1} t_k]$$

or

$$0 \le t_h, \quad ht_h + \sum_{k=1}^{h-1} t_k \le 1 \qquad (h = 1, \ldots, n). \tag{12}$$

Each of these inequalities determines a closed half-space. The set K of points $(r_1, \ldots, r_n)$ in $\mathbb{R}^n$ satisfying all of these inequalities (with r_k in place t_k) has $r_h \leq \frac{1}{h}$. Thus K is a compact, convex subset of $\mathbb{R}^n$.

Without taking into account whether or not there is a deterministic strategy corresponding to the numbers $s_1, \ldots, s_n$, if we consider the problem of maximizing (10) by choices of s_h satisfying (11), it translates to the problem of maximizing $\sum_{h=1}^{n} ht_h$ by choices of t_h satisfying (12). Now, the possible values for s_h form a finite set so that the possible points $(t_1, \ldots, t_n)$ in K constitute a finite subset. Nonethess, we shall study the question of maximizing the linear functional

$$f: \quad (r_1, \ldots, r_n) \to \sum_{h=1}^{n} hr_h$$

on K. Of course f attains its maximum on (a closed face of the compact convex) K (and at an extreme point).

Suppose that $(t_1, \ldots, t_n)$ is a point in K at which f attains its maximum. We can assume that n is 3 or greater, for the question of best strategy is trivial when n is 1 or 2. We note that the point $(1, 0, \ldots, 0)$ is not a maximizing point since $(0, \frac{1}{2}, \frac{1}{6}, 0, \ldots, 0)$ lies in K and

$$f(1, 0, \ldots, 0) = 1 < \tfrac{3}{2} = f(0, \tfrac{1}{2}, \tfrac{1}{6}, 0, \ldots, 0).$$

We prove the following assertion.

(*) *If $t_k = 0$ for some k, then $kt_k + \sum_{j=1}^{k-1} t_j < 1$.*

It is true when k is 1. If $t_2 = 0$ and $2t_2 + t_1 = 1$, then $t_2 = \cdots = t_n = 0$ since $kt_k + \sum_{j=1}^{k-1} t_j \leq 1$. But we have just noted that $(1, 0, \ldots, 0)$ is not a maximizing point for f. Thus (*) holds when k is 2. Assume, now, that we have proved (*) for all values less than $k+1$. Suppose $k \geq 2$ and $t_{k+1} = 0$. If

$$1 = (k+1)t_{k+1} + \sum_{j=1}^{k} t_j = \sum_{j=1}^{k} t_j,$$

then

$$kt_k + \sum_{j=1}^{k-1} t_j = (k-1)t_k + \sum_{j=1}^{k} t_j = (k-1)t_k + 1 \leq 1.$$

Since $1 \leq k-1$, we must have that $t_k = 0$, whence

$$1 = \sum_{j=1}^{k} t_j = kt_k + \sum_{j=1}^{k-1} t_j,$$

contradicting our inductive hypothesis. Thus

$$(k+1)t_{k+1} + \sum_{j=1}^{k} t_j < 1,$$

and $(*)$ is valid for all k.

We show, next, that $t_j = 0$ for $j \leq k$ if $t_k = 0$. Suppose $t_k = 0$ so that

$$\sum_{j=1}^{k-1} t_j = kt_k + \sum_{j=1}^{k-1} t_j < 1.$$

If $t_h > 0$ for some h in $\{1, \ldots, k-1\}$, choose ε positive and less than t_h and $k^{-1}[1 - \sum_{j=1}^{k-1} t_j]$. Let $(u_1, \ldots, u_n)$ be the result of replacing t_h and t_k in $(t_1, \ldots, t_n)$ by $t_h - \varepsilon$ and ε, respectively. Then $0 \leq u_j$ for all j and

$$pu_p + \sum_{j=1}^{p-1} u_j \leq pt_p + \sum_{j=1}^{p-1} t_j \leq 1$$

when $p \in \{1, \ldots, n\}$. Thus $(u_1, \ldots, u_n) \in K$. But

$$f(t_1, \ldots, t_n) = \sum_{j=1}^{n} jt_j < \sum_{j=1}^{n} ju_j = \sum_{j=1}^{n} jt_j + (k-h)\varepsilon = f(u_1, \ldots, u_n),$$

contradicting the choice of $(t_1, \ldots, t_n)$ as maximizing. Thus $t_j = 0$ for $j \leq k$ when $t_k = 0$, as asserted.

Assume, now, that $t_1 = t_2 = \cdots = t_k = 0$ and $t_{k+1} \neq 0$. From what we have just proved, $t_{k+2}, \ldots, t_n$ are all positive. Suppose that for some r greater than k

$$(r+1)t_{r+1} + \sum_{j=1}^{r} t_j < 1.$$

Choose ε positive but smaller than $r^{-1}[1 - (r+1)t_{r+1} - \sum_{j=1}^{r} t_j]$ and t_r. Let $(v_1, \ldots, v_n)$ be the result of replacing t_r and t_{r+1} in $(t_1, \ldots, t_n)$ by $t_r - \varepsilon$

and $t_{r+1}+\varepsilon$, respectively. Then $v_j \geq 0$ for each j and

$$pv_p + \sum_{j=1}^{p-1} v_j \leq pt_p + \sum_{j=1}^{p-1} t_j \leq 1$$

when $p \in \{1,\dots,r,r+2,\dots,n\}$. Thus $(v_1,\dots,v_n) \in K$. But

$$f(t_1,\dots,t_n) = \sum_{j=1}^{n} jt_j = \sum_{j=1}^{n} jv_j - \varepsilon < \sum_{j=1}^{n} jv_j = f(v_1,\dots,v_n),$$

contradicting the choice of $(t_1,\dots,t_n)$ as maximizing. We conclude from this that $(r+1)t_{r+1} + \sum_{j=1}^{r} t_j = 1$ for all r in $\{k+1,\dots,n\}$.

The equalities

$$(13) \qquad t_1 = \cdots = t_k = 0, \quad rt_r + \sum_{j=1}^{r-1} t_j = 1 \qquad (r = k+2,\dots,n)$$

imply that

$$rt_r = 1 - \sum_{j=1}^{r-1} t_j, \quad (r+1)t_{r+1} = 1 - \sum_{j=1}^{r} t_j = rt_r - t_r = (r-1)t_r.$$

Thus $t_{r+1} = \frac{r-1}{r+1} t_r$, from which

$$\begin{aligned} t_r &= \frac{r-2}{r} t_{r-1} = \frac{(r-2)(r-3)}{r(r-1)} t_{r-2} \\ &= \cdots = \frac{(r-2)(r-3)\cdots(k+1)}{r(r-1)\cdots(k+3)} t_{k+2} \\ &= \frac{(k+2)(k+1)}{r(r-1)} t_{k+2} = \frac{k+1}{r(r-1)}(1 - t_{k+1}) \quad (r = k+2,\dots,n). \end{aligned}$$

Hence the maximizing point $(t_1,\dots,t_n)$, lies on the line segment in $\mathbb{R}^n$ whose points $(x_1 \dots, x_n)$ satisfy

$$\begin{aligned} &x_1 = \cdots = x_k = 0, \\ &x_r = \frac{(k+1)}{r(r-1)}(1 - x_{k+1}) \qquad (r = k+2,\dots,n), \\ &0 \leq x_{k+1} \leq \frac{1}{k+1}. \end{aligned}$$

The maximizing point for f (assigning $\sum_{j=1}^{n} jx_j$ to points $(x_1 \ldots, x_n)$ in $\mathbb{R}^n$) occurs on this segment (at an endpoint, and 0 is proscribed by our assumption that $t_{k+1} \neq 0$) when x_{k+1} is $\frac{1}{k+1}$. Thus $t_1 = \cdots = t_k = 0$, $t_{k+1} = \frac{1}{k+1}$, and $t_r = \frac{k}{r(r-1)}$. Thus

$$\sum_{j=1}^{n} jt_j = \sum_{j=k+1}^{n} jt_j = \sum_{j=k+1}^{n} \frac{k}{j-1} = k\sum_{j=k}^{n-1} \frac{1}{j}.$$

The values t_h just determined (in terms of k) arise from a deterministic strategy S_k with s_h elements of length h, where $s_1 = \cdots = s_k = 0$ and

$$s_h = h!t_h = k(h-2)! \qquad (h = k+1, \ldots, n)$$

that produces $(n-1)!\sum_{j=1}^{n} jt_j = n!\frac{k}{n}\sum_{j=k}^{n-1} \frac{1}{j}$ wins. The strategy S_k rejects the first k elements drawn then accepts anyone of the $k!$ top elements on the $k+1$ st draw, should one appear, then any of the top elements that do not have their initial segments of length $k+1$ top elements appearing on the $k+2$ nd draw, and so forth. In other words, S_k is the strategy that rejects the first k elements drawn and then accepts the next element drawn larger than all of those.

We want to find k in $\{1, \ldots, n-1\}$ that maximizes $n!\frac{k}{n}\sum_{j=k}^{n-1} \frac{1}{j}$, equivalently, that maximizes $\frac{k}{n}\sum_{j=k}^{n-1} \frac{1}{j}$; our interest is in maximizing this expression for "large" n. By forming the Riemann approximating sum to $\int_{\frac{k}{n}}^{1} \frac{dy}{y}$ based on the partition of $[\frac{k}{n}, 1]$ into subintervals of length $\frac{1}{n}$ and evaluating $\frac{1}{y}$ first at all the left endpoints and then at all the right endpoints of the partition subintervals, we conclude that

$$\sum_{j=k+1}^{n} \frac{1}{j} \leq \int_{\frac{k}{n}}^{1} \frac{dy}{y} \leq \sum_{j=k}^{n-1} \frac{1}{j}.$$

Thus

$$0 \leq \frac{k}{n}\sum_{j=k}^{n-1} \frac{1}{j} - \frac{k}{n}\int_{\frac{k}{n}}^{1} \frac{dy}{y} \leq \frac{k}{n}\left(\frac{1}{k} - \frac{1}{n}\right) \leq \frac{1}{n}.$$

It follows that the choice of k that maximizes $\frac{k}{n}\int_{\frac{k}{n}}^{1} \frac{dy}{y}$ puts $\frac{k}{n}\sum_{j=k}^{n-1} \frac{1}{j}$ to within $\frac{1}{n}$ of its maximum. Writing x for $\frac{k}{n}$, the maximum for $-x\log x$ ($= x\int_{x}^{1} \frac{dy}{y}$) on $[\frac{1}{n}, 1]$ occurs when x is $1/e$ and that maximum is $1/e$. This value of x corresponds to choosing n/e for k and the strategy produces more than

$n!/e$ wins. If we alter this choice of k replacing it by the nearest integer, x changes by less than $\frac{1}{n}$ and $-x \log x$ changes by less than e/n. This would change the number of wins on the order of $(n-1)!$ which is an insignificant fraction of the total number $n!$ of games. Thus, asymptotically, the best deterministic strategy is to reject the first (approximately) n/e elements drawn and then to accept the next draw larger than all of those (if there is such and, otherwise, proceed to end of the game — with a loss).

5. **Postscript.** My thanks are due to Ronald Graham and Mogens Hansen for reading the manuscript; each noted a small error (both, hopefully, corrected). Special thanks are due, again, to Ronald Graham for supplying me with copies of several articles, published over the years, on the secretary problem. I had heard this name for the problem some ten years ago from a junior colleague (after talking casually about various items of this nature over lunch), but he gave me no references. Apparently there is a vast and lively literature on the subject. The earliest allusion to the problem is to a statement of it by Andrew Gleason in 1955 followed by his comment that he had heard it elsewhere. He had *not* heard it from me, but during the summer of 1955, at a summer meeting of the AMS, Gleason and I discussed the problem at an evening gathering (we have been friends since 1950). He mentioned it to me asking me what I thought the answer might be. I teased him by musing for a few seconds and then remarking that you should probably reject about one third and then choose the next highest. I cannot remember whether I left him "amazed" (and silent) at my speed (!) or told him the full (Columbia) story.

My intention was to include no references; I knew none. I now know too many. Since referencing is not really pertinent to this article, I shall include just two serious, general references from which the interested reader can gain access to the literature.

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